

On the consistency of $P = NP$ with fragments of ZFC whose own consistency strength can be measured by an ordinal assignment.*

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Abstract

We formulate the $P < NP$ hypothesis in the case of the satisfiability problem as a Π_2^0 sentence, out of which we can construct a partial recursive function $f_{\neg A}$ so that $f_{\neg A}$ is total if and only if $P < NP$. We then show that if $f_{\neg A}$ is total, then it isn't \mathcal{T} -provably total (where \mathcal{T} is a fragment of ZFC that adequately extends PA and whose consistency is of ordinal order). Follows that the negation of $P < NP$, that is, $P = NP$, is consistent with those \mathcal{T} .

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1 Introduction

As it is well-known, Gentzen's proof of the consistency of arithmetic (PA) requires a transfinite induction up to ϵ_0 . Ackermann's proof of the consistency of the theory of real numbers (RT) asks for an induction up to the first η number, that is, the first ϵ -number κ such that $\epsilon_\kappa = \kappa$. Wainer remarked (personal communication; see also [14, 15]) that some results by Kreisel [10] can be extended to fragments of ZFC whose own consistency is measured by an ordinal, as in the cases of PA and RT. Therefore, it is possible to show in this way that certain sentences are not provable in those fragments and, as a consequence, that the negations of such sentences are consistent with them.

We apply those ideas to prove that $P = NP$, in the particular case of the satisfiability problem for Boolean expressions, is consistent with PA and other fragments of ZFC whose consistency can be measured by an adequate recursive ordinal. It immediately follows that $P = NP$ in its general form is also consistent with those theories.

Our exposition will be somewhat informal [12] and intuitive; however it is easy to reformulate it in a rigorous way.

Summary of the paper

1. We start from PA and from the satisfiability problem \mathcal{S} for Boolean expressions in conjunctive normal form (cnf).

We note $[P = NP]^\mathcal{S}$ the sentence that asserts that there is a polynomial algorithm that settles all instances of the satisfiability problem.

2. We obtain the function $f_{\neg A}$. $f_{\neg A}$ is total if and only if $\neg[P = NP]^\mathcal{S}$ (that is, $[P < NP]^\mathcal{S}$) holds.
3. We show that $f_{\neg A}$ lies "beyond" the Kreisel hierarchy. Then:
 - (a) If $f_{\neg A}$ is total in the standard model for PA, then by a theorem of Kreisel [10] PA doesn't prove $\neg[P = NP]^\mathcal{S}$ and therefore $[P = NP]^\mathcal{S}$ is consistent with PA.
 - (b) If $f_{\neg A}$ isn't total in the standard model for PA, then again $[P = NP]^\mathcal{S}$ is consistent with PA, if we suppose that PA only proves sentences that are true of the standard model.
 - (c) Follows that $[P = NP]^\mathcal{S}$ is consistent with PA.

4. Of course this result extends to all other problems in the NP -class, so that we have that $[P = NP]$ in general is consistent with PA.
5. The proof given for PA is also valid for any theory that "includes" PA and whose consistency can be measured by an adequate recursive ordinal. (We moreover suppose that the theory is adequately sound.)

2 $[P < NP]^S$ as a Π_2^0 sentence

Remark 2.1 Suppose given the canonical enumeration of binary words

$$\emptyset, 0, 1, 00, 01, 10, 11, 000, 001, \dots$$

which code the empty word and the integers

$$\sqcup \text{ (blank)}, 0, 1, 2, \dots$$

We take this correspondence to be fixed for the rest of this paper. \square

Remark 2.2 We consider cnf-Boolean expressions, that is, Boolean expressions in conjunctive normal form.

- Let x be a Boolean expression in cnf, adequately coded as a binary string of length $|x|$. Let P_n be a polynomial machine of Gödel number n . (We show below in Section 3 how to construct a recursive sequence of polynomial machines that suits our purposes in this paper.)
- Given a string y of truth-values for the $|y|$ Boolean variables of x , there is a polynomial procedure (a polynomial Turing machine which we will note V) that will test whether y satisfies x , that is, say, $V(\langle x, y \rangle) = 1$ if and only if y satisfies x ; and equals 0 otherwise.

$\langle \dots, \dots \rangle$ is the usual [12] pairing function; we will only write it when required to avoid ambiguity.

(For the sake of completeness, we add that $V(0, 0) = 1$, that is, the empty string is satisfied by the empty string.)

- We formulate the recursive predicate:

$$A^*(m, x) \leftrightarrow_{\text{Def}} \exists y (P_m(x) = y \wedge V(x, y) = 1).$$

$A^*(m, x)$ is intuitively understood as “polynomial machine of index m accepts Boolean cnf expression x ,” that is, “machine m inputs x and outputs a line of truth values that satisfies x .”

- We can also write: $A^*(m, x) \leftrightarrow V(x, P_m(x)) = 1$.
- Form the pair $z = \langle x, y \rangle$, and let π_i , $i = 1, 2$, be the usual (polynomial) projection functions. Recall that V is a polynomial machine that inputs a pair $\langle x, y \rangle$. Then we can consider the predicate:

$$\neg A(m, z) \leftrightarrow_{\text{Def}} V(z) = 1 \wedge V(\langle \pi_1 z, P_m(\pi_1 z) \rangle) = 0,$$

or

$$\neg A(m, z) \leftrightarrow V(z) = 1 \wedge \neg A^*(m, \pi_1 z).$$

- $A(m, z)$ can be intuitively read as follows: polynomial machine P_m doesn't accept the pair z if and only if z is such that $\pi_1 z = x$ is satisfiable, but the output of P_m over $x = \pi_1 z$ doesn't satisfy x .

This form for $\neg A(m, x)$ was suggested by F. Cucker to the authors. \square

Proposition 2.3 $[P < NP]^S \leftrightarrow \forall m \exists z \neg A(m, z)$. \square

Remark 2.4 Notice that if $[P < NP]^S$ holds, then the existence of a single z_0 such that $\neg A(m, z_0)$ implies that there are infinitely many z'_0 such that $\neg A(m, z'_0)$. \square

The function $f_{\neg A}$

Definition 2.5 $f_{\neg A}(m) =_{\text{Def}} \mu_x \neg A(m, x)$. \square

Lemma 2.6 $[P < NP]^S \leftrightarrow f_{\neg A}$ is total. \square

PA-provably total recursive functions

The concept we now introduce originated in [10] Kreisel: we say that a recursive function f is PA-provably total unary recursive if, for some Gödel number e :

1. PA proves that e is the Gödel number of f , and
2. For each x , PA proves that the computation of $f(x)$ converges.

In what follows we suppose that all variables are restricted to ω , the set of natural numbers. Formally,

Definition 2.7 A PA-unary function $f : \omega \rightarrow \omega$ is PA-provably total unary recursive if for some Gödel number e_f for f ,

$$\text{PA} \vdash \forall x \exists z (T(e_f, x, z) \wedge \forall y (f(y) = \{e_f\}(y))). \square$$

Therefore $T(e_f, x, z)$ holds, and there is a z so that the computation of e_f over x stops in z steps, and not before, for every x . This means that we have a proof in PA that every computation of f converges.

Remark 2.8 Some of those non-PA-provably total unary functions 'top' all PA-provably total unary functions. \square

Definition 2.9 For $f, g : \omega \rightarrow \omega$,

$$f \text{ dominates } g \leftrightarrow_{\text{Def}} \exists y \forall x (x > y \rightarrow f(x) \geq g(x)).$$

We write $f \succ g$ for f dominates g . \square

We need the next (trivial) result:

Corollary 2.10 *If, for any PA-provably total recursive unary function f we have that g overshoots through f infinitely many times (that is, for infinitely many x , $g(x) > f(x)$), then g isn't a PA-provably total unary recursive function.* \square

3 Polynomial Turing machines

Remark 3.1 We describe the behavior of the Turing machines we consider here to avoid ambiguities:

1. Turing machines are defined over the set A_2^* of finite words on the binary alphabet $A_2 = \{0, 1\}$.
2. Each machine has $n + 1$ states s_0, s_1, \dots, s_n , where s_0 is the final state. (The machine stops when it moves to s_0 .)
3. The machine's head roams over a two-sided infinite tape.
4. Machines input a single binary word and either never stop or stop when the tape has a finite, and possibly empty set of binary words on it.
5. The machine's *output word* will be the one over which the head rests if and when s_0 is reached. (If the head lies on a blank square, then we take the output word to be the empty word.) \square

Remark 3.2 Our Turing machines input a binary string $[x]$ and output a binary string $[y]$ that stands for the numeral y . The corresponding recursive functions input the numeral x and output the numeral y . However as it will be always clear from context, we write x for both the binary sequence and the numeral. \square

Remark 3.3 We emphasize that the computable functions we are dealing with are always given as Turing machines. We will use upper-case sans serif letters (M, \dots) for the machines. If M_n is a Turing machine of Gödel number n , its input-output relation is noted $M_n(x) = y$. \square

Remark 3.4 We define the *empty* or *trivial machine* to be the Turing machine with an empty table; we take it to be the simplest example of the identity machine, again by definition. \square

Gödel numbering

Remark 3.5 Turing machines are code lines ξ, ξ', \dots , separated by blanks \sqcup , such as $\xi \sqcup \xi' \sqcup \dots \sqcup \xi''$. Let Ξ be one such set of code lines separated by blanks. Let Ξ' be obtained out of Ξ by a permutation of the lines ξ, ξ', \dots . Both Ξ and Ξ' are seen as different machines that compute the same algorithmic function, that is, in this case, if $f_\Xi (f_{\Xi'})$ is computed by $\Xi (\Xi')$, then $f_\Xi = f_{\Xi'}$. \square

We allow some freedom in the construction of those ‘code lines’ $\xi \dots$. Usual program instructions are acceptable.

Remark 3.6 We can recursively enumerate all tables for Turing machines as described in Remarks 3.1 and 3.5. We suppose that our Gödel numbering for Turing machines arises out of the following:

- We list all words in the alphabet used to describe the Turing machine tables.
- The set of tables is recursive, and so, given a binary word x ,
 1. If it translates into a Turing machine table through the usual 1–1 correspondence between words and machines, then the corresponding numeral codes that machine.
 2. If not, we impose that it will code the trivial machine. \square

Remark 3.7 We can explicitly construct Turing machines F_α , $0 \leq \alpha < \epsilon_0$ for the Kreisel hierarchy of fast-growing total recursive functions f_α , which are used here according to the prescriptions in [10, 13, 14, 15]. \square

We present our arguments and constructions for PA, that is, for the machines $F_0, F_1, \dots, F_\alpha, \dots$, $\alpha < \epsilon_0$. Extension to other theories whose consistency is of ordinal rank will be discussed in the course of our argument.

Machines bounded by a clock

In order to deal with the set of all polynomial machines we must resort to a trick. That is, we deal with a recursive set of *expressions* for polynomial machines, so that for each “concrete” polynomial machine there is an expression for that machine, and no expression in the listing represents a nonpolynomial machine.

Remark 3.8 The idea goes as follows:

1. We use at first a variant of the [1] Baker–Gill–Solovay trick: we consider all couples $\langle M_n, C_p \rangle$ where M_n is a Turing machine, and C_p is a polynomial clock (Definition 3.10) that, for an input x of length $|x|$, shuts down the machine after $C_p(|x|)$ steps, if the machine is still running [1].

We are soon going to specify the structure of the clocks we use; they will always be PA-provably total Turing machines.

2. Since there is an uniform recursive procedure to obtain a Turing machine out of each pair $\langle M_n, C_p \rangle$, we ‘embed’ this recursive sequence of pairs (that represent polynomial machines) into the sequence of all Turing machines.
3. Notice that each polynomial machine will be ‘represented’ by several Turing machines.
4. We will take the Turing machine Gödel number as the Gödel number for our polynomial machines.

In order to handle the machine pairs, we define below an adequate indexing system for them that includes both the machine and the clock that bounds its operation time. \square

Parametrized polynomial clocks

Remark 3.9 We suppose that the clock acts as follows. If $M_n(x)$ stops before the clock determines it to shut down, the output is precisely $M_n(x)$. However if the bound in the number of processing steps is reached before $M_n(x)$ stops, the clock stops it and orders its state to move to s_0 . The machine’s output is then agreed to be 0. \square

Definition 3.10 A parametrized polynomial clock $C_{F(k)}$ is a total Turing machine that depends on a positive integer k and on a total recursive F , and that satisfies:

1. The clock is coupled to another Turing machine M_n .
2. Whenever x is input to M_n , x is also input to $C_{F(k)}$.
3. $C_{F(k)}$ computes $|x|^{F(k)} + F(k)$.
4. If $M_n(x)$ hasn’t stopped before, then $C_{F(k)}$ shuts it down after it has completed $|x|^{F(k)} + F(k)$ steps in the computation over x and makes it output 0. \square

Remark 3.11 We will restrict our attention to the following class of clocks that will be coupled to the Turing machines in order to make them polynomial machines:

1. C_p is the clock that shuts down the Turing machine M_n in the machine pair $\langle M_n, C_p \rangle$ after $|x|^p + p$ steps, as described above.
2. Now consider the Kreisel hierarchy [7, 10, 14]. For the ordinal indices

$$\omega, \omega^\omega, \omega^{\omega^\omega}, \dots < \epsilon_0,$$

in its obvious enumeration, form the corresponding functions F_0, F_1, \dots , where ‘0’ stands for ω , ‘1’ stands for ω^ω , and so on.

3. Then $C_{F_\alpha(k)}$ denotes the clock that shuts down the Turing machine in the manner prescribed above after $x^{F_\alpha(k)} + F_\alpha(k)$ steps, where F_α is indexed as described.
4. We will consider two classes of clocks: those that correspond to the polynomials $|x|^p + p$, $p = 0, 1, 2, \dots$, and the (families of) clocks that correspond to the parametrized polynomials $x^{F_\alpha(k)} + F_\alpha(k)$.
5. We suppose given a fixed recursive enumeration for those two kinds of clocks.

This restriction won't affect our result. \square

In what follows we will only deal with those two kinds of clocks.

Remark 3.12 For the case where we are dealing with a theory \mathcal{T} of consistency rank $\zeta > \epsilon_0$, recall that ζ is a constructive ordinal. Therefore we can always find a recursive subset of ordinals as those in step 2 above to proceed as stated. \square

Remark 3.13 Again let $\langle \dots, \dots \rangle$ denote the usual [12] degree-2 polynomial recursive 1-1 pairing function $\langle \dots, \dots \rangle : \omega \times \omega \rightarrow \omega$. \square

We define:

Definition 3.14 $P_p = \langle M_i, C_j \rangle$, C_j a parametrized clock. Call their set \mathcal{P} , the set of Turing machines coupled to parametrized polynomial clocks. \square

Remark 3.15 \mathcal{P} contains all possible ordered pairs as in the preceding definitions; the number p can be intuitively seen as coding an expression for a Turing machine with a clock. In what follows \mathcal{M} will denote the sequence of all Turing machines. \square

Proposition 3.16 There is a recursive 1-1 embedding $\sigma : \mathcal{P} \rightarrow \mathcal{M}$ of the polynomial machines represented by pairs $\langle M_m, C_n \rangle$, into polynomially-bounded Turing machines given by their tables. The set $\sigma\mathcal{P}$ is recursive in \mathcal{M} . \square

Remark 3.17 Suppose that we are given the image $\sigma\mathcal{P}$ of that map into \mathcal{M} , and suppose moreover that we add to that image all explicitly defined polynomial machines that we use in this construction, such as the ones of the form given in Lemma 3.21 below. Those machines are finite-output machines (in a sense made clear in that lemma); we note their set \mathcal{F} .

The resulting set $\mathcal{F} \cup \sigma\mathcal{P}$ of polynomial machines is recursive, and the consistency result we describe here related to $f_{\neg A}$ is valid if and only if it is valid when extended both to that subset $\sigma\mathcal{P} \subset \mathcal{M}$ when coded by their own Turing-machine Gödel numbers and to $\mathcal{F} \cup \sigma\mathcal{P}$. \square

Remark 3.18 We form an extended f' by defining it to be equal to f over $\sigma\mathcal{P}$ and 0 otherwise. Therefore $f'_{\neg A}$ is total if and only if the extension $f'_{\neg A}$ is total. \square

Since there is such a recursive, uniform procedure σ that allows us to obtain actual tables for polynomial Turing machines given an arbitrary machine and clock as described here, we can form their compositions, which are again polynomial machines, granted a recursive rule to define a bounding clock.

We can easily obtain, out of an adequate definition for the composition operation, that:

Proposition 3.19 \mathcal{P} is closed under composition, that is, if $P, P' \in \mathcal{P}$, then we can obtain a $P'' \in \mathcal{P}$ such that $P'' = P \circ P'$ as a function. \square

Two side comments

(See [11].) Polynomial Turing machines can be very powerful:

Remark 3.20 We first show that if G is a fast-growing (“hard,” “intractable”) recursive function that is the most efficient way of settling a given problem, and if H is another such function, then there is a family of polynomial Turing machines parametrized by $n \in \omega$ such that machine n settles exactly $H(n)$ instances of the problem we are dealing with:

Lemma 3.21 Let n be any fixed natural number and let G, H be Turing machines that compute two arbitrary, monotonic increasing, total unary recursive functions. Then for $P_{(G,H)}$ given by:

1. $P_{(G,H)}(x) = G(x)$, $x \leq H(n)$.
2. $P_{(G,H)}(x) = 0$, $x > H(n)$.

there is a polynomial algorithm for it.

Sketch of proof: We can construct the table of P in such a way that the operation time of P is bounded by a constant larger than the amount of time required for the largest computation of $G(x)$, $x \leq H(n)$. \square

(We suppose that 0 is no solution for the problem which G settles, for $x > 0$.) So, we have established the existence of the family that we are looking for. \square

An interesting intractable problem for polynomial machines is described below:

Lemma 3.22 It is “hard” to determine, for an arbitrary n , the value of $H(n)$, which is the cardinality of the set of instances of the desired problem which are settled by the n -th machine in that family. \square

Definition 3.23 The set of all machines as in Lemma 3.21 is noted \mathcal{F} . \square

4 Sketch of the main result

Remark 4.1 From here on we act according to the following:

- We have added to the (recursive) image $\sigma\mathcal{P} \subset \mathcal{M}$ all machines in Remark 3.20 besides the images of the pairs $\langle M, C \rangle$ to get $\mathcal{F} \cup \sigma\mathcal{P}$.
- That $f_{\neg A}$ has been extended to $f'_{\neg A}$ over \mathcal{M} as in Remarks 3.17 and 3.18. However by an abuse of language we will use $f_{\neg A}$ for the extended function. \square

We can now summarize our main argument.

Remark 4.2 Recall that if f is PA-provably total recursive, then there is a PA-provably total recursive g such that $g \succ f$.

$f_{\neg A}$ will be seen not to be PA-provably total. \square

Remark 4.3 We add more detail to our argument. Suppose that we have constructed $\sigma\mathcal{P} \subset \mathcal{M}$, and that $f_{\neg A}$ has been extended as indicated above to the whole of \mathcal{M} :

- Let F be a Turing machine that computes a recursive, unary total function which dominates all PA-provably total functions. We can take $F = F_{\epsilon_0}$ in one of the hierarchies described in [7, 10, 13, 14].
- Let E denote a fixed, explicitly given exponential Turing machine that solves any instance of the satisfiability problem.
- For each natural number n , form a polynomial machine $Q^{F(n)}$ that operates as follows:
 1. Put $K = F(n)$.
 2. For $x \leq K$, $Q^{F(n)}(x) = E(x)$.
 3. For $x > K$, $Q^{F(n)}(x) = 0$.

(Notice that 0 is never a solution, for $x > 0$.) So up to $F(n)$ the machine $Q^{F(n)}$ equals $E(x)$. From then on it always prints 0.

This machine $Q^{F(n)}$ is polynomial (see Lemma 3.21).

- Its Turing machine table can be described out of the following set of instructions:
 1. The code for the parameter n . (Perhaps a single instruction line, as $y = n$.)
 2. The code for computing F . (Result of the computation should be $F(y)$, for y as input.)

3. Instructions for the computation of $Q^z(x)$, for $z = F(y)$. (Here the instructions involve the variables x and y .) This involves the fixed code for E .

(See in Remark 3.5 the coding procedures for Turing machines that we use in this paper.)

- Notice that (if we use the coding techniques described in Remark 3.5), for each n , there are constant natural numbers a, b such that the Gödel number for $Q^{F(n)}$ equals $an + b$. (See Proposition 5.9.)
- The operation time of $Q^{F(n)}$ is bounded by a polynomial clock with bound $F'(n) + x^{F'(|n|)}$, each n , where $|x|$ is the length of the binary word x , for sufficiently large $F' > F$. (It is enough to take $F' = F_{\epsilon_0+1}$.)
- Again the instructions for the operation of that clock are such that, given each n , their Gödel numbers are given by $a'n + b'$. (See Proposition 5.9.)
- The pairing $\langle an + b, a'n + b' \rangle$ is quadratic on n [12].
- Thus (intuitively) $f_{\neg A}(\langle an + b, a'n + b' \rangle) = F(n) + 1$ (over the machine pairs), or $f_{\neg A}(an + b) = F(n) + 1$ over $\mathcal{F} \cup \sigma\mathcal{P}$. Intuitively again, our function $f_{\neg A}$ overshoots infinitely many times through every provably total unary recursive function in PA.

Given the preceding results, plus Lemma 5.17, we conclude that $f_{\neg A}$ isn't dominated by any such PA-provable total recursive function.

See Remark 5.28 in the final Section of the paper. \square

Remark 4.4 We can also argue as follows: instead of using the “ceiling function” F_{ϵ_0} , let us be given the Kreisel [7, 10] hierarchy $\{F_0, F_1, \dots, F_\beta, \dots\}$, $0 \leq \alpha < \epsilon_0$, of PA-provably total functions and let F_α be in that hierarchy. Then:

- We make the preceding construction for a F_β such that $F_\beta \succ F_\alpha$.
- This will be true even if F_α is composed with a quadratic function of its variable, that is, $F_\beta \succ F_\alpha \circ u$, where u is quadratic, and so on.
- This can be repeated for any α in the Kreisel hierarchy.
- So, we conclude that no PA-provably total recursive function can dominate $f_{\neg A}$. \square

Remark 4.5 Since we will work within the set of Turing machines \mathcal{M} , as given in Proposition 3.16 the Gödel number for the machines Q will be a linear function $an + b$. \square

Remark 4.6 We do not obtain this result if we restrict ourselves to an enumeration such as the one in [1]. (We thank S. Wainer for that observation.)

Yet we feel it isn't natural to exclude polynomial Turing machines such as those in our family $Q^{F(n)}$, any n (Example 3.20) since that family contains rather simple polynomial machines with the property that it is very hard to compute the input values at which they start to output zeros forever.

The gist of the matter is this fact: a set of machines (given by their tables) can only contain polynomial Turing machines, but it may be very, very hard to decide some of its properties. Our family $Q^{F(n)}$, all n , is an example of such a set. \square

5 Computation of some infinite segments of the function $f_{\neg A}(n)$

We can restrict our attention to the machines in \mathcal{F} . We will argue for PA, but it is easy to extend our results to the \mathcal{T} already characterized. We recall:

Definition 5.1 $f_{\neg A}(m) =_{\text{Def}} \mu_x \neg A(m, x)$. \square

Remark 5.2 Now either “ $f_{\neg A}$ is total” is true of the standard model or it isn't. \square

We prove here:

Proposition 5.3 *Given any F_α , $0 \leq \alpha < \epsilon_0$ then for no α does F_α dominate $f_{\neg A}$.*

Corollary 5.4 *If PA is arithmetically sound, then $PA \not\models f_{\neg A}$ is total. Therefore $[P = NP]^S$ is consistent with PA.* \square

Proof: Will be given throughout this Section.

Remark 5.5 For “arithmetic soundness” (or “arithmetic consistency”) see [3]. \square

Computation of the Gödel number of $Q^{F_\alpha(n)}$

Remark 5.6 We must clearly separate the two kinds of machine codes that we will be using for the purposes of our proof:

- The Gödel number of a Turing machine is its \mathcal{M} -code, index or simply its Gödel number.

- We also code our polynomial machines out of the Gödel numbers of the paired machine-and-clock $\langle M, C \rangle$ representation. This specific code for polynomial machines is their \mathcal{P} -index or code. \square

Remark 5.7 The argument presented in this subsection is very simple. Gödel numbers for the machines in our proof will turn out to be as in an arithmetic progression like 547, 647, 747, 847, \dots

To add some more detail:

- *Gödel numbers.* We show that the Gödel numbers for the machines that interest us and the corresponding clocks can be written as a string $[n]s$ ($n = 0, 1, 2, \dots$), $[n]$ is a binary string that includes the binary string for n , and s is a fixed binary string; concatenation of both strings is indicated by their juxtaposition. As we noticed, those Gödel numbers are in an arithmetic progression.
- *Indices for machine pairs.* As the usual pairing function [12] $\langle x, y \rangle$ is a degree-2 polynomial, the code for a pair $\langle M, C \rangle$ will depend on a degree-2 polynomial on n when parametrized as above. \square

Our goal is to compute the \mathcal{M} - and \mathcal{P} -indices of the machines $Q^{F_\alpha(n)}$ as a function of the tables for F_α , for those finite machines described above, and as a function of n .

For notation see Remarks 3.5 and 3.6. We consider machines $Q^{F_j(n)}$ and the pairs $\langle Q^{F_j(n)}, C \rangle$, with a clock C that bounds the polynomial machine without cutting it off before it stops by itself.

We will first deal with the indexing over \mathcal{M} , and then over \mathcal{P} . See Remark 3.6 and recall that “garbage” is mapped on the trivial machine.

Remark 5.8 For the remainder of this Section, Turing machine tables will be given in the form of sketchy, summary programs. \square

Estimates for Gödel numbers

Our result on Gödel numbers is:

Proposition 5.9 *We can write the table for $Q^{F_\alpha(n)}$, each n , so that its Gödel number (\mathcal{M} -index) $\rho(n, \alpha) = a_\alpha + (2^{q_\alpha} - 1)n$, a_α, q_α positive integers, q_α a constant that depends on the Gödel number for F_α .*

Example 5.10 We can give an example to make things explicit. The (sketchy) program we use will look like:

1. Start.

2. $y = n$.
3. Input program for F_α .
Compute $F_\alpha(y)$.
4. Input program for 2^x .
Compute 2^z , for $z = F_\alpha(y)$.
5. Output $2^z + 1$.
6. Stop.

Proof: Recall that $\lfloor x \rfloor$ is a binary string for x . We may write the program for the polynomial Turing machine $Q^{F_\alpha(n)}$ as the concatenation $\lfloor [n] \rfloor \lfloor \xi'_\alpha \rfloor \lfloor \xi'' \rfloor$, where:

- $\lfloor [n] \rfloor$ codes lines 1 and 2.
It is a binary string that includes the bits for the numeral n (that is the only variable portion in the machine's program).
- ξ'_α codes line 3; essentially the program for F_α ;
- ξ'' codes lines 4 and 5.

Given our Gödel numbering conventions, the value of the Gödel number for $Q^{F_\alpha(n)}$, for $q_\alpha = \lfloor \xi'_\alpha \xi'' \rfloor$, is given by a binary number $\rho(n, \alpha) = \lfloor [n] q_\alpha \rfloor$.

Machines $Q^{F_\alpha(n)}$, for each n , as given by the tables so described are regularly spaced among the M_m in the ordering we have given for \mathcal{M} (see Remark 3.6), that is, at most $2^{q_\alpha} - 1$ machines lie between machine $Q^{F_\alpha(n-1)}$ and machine $Q^{F_\alpha(n)}$ in that arrangement. In other words, the Gödel number of a $Q^{F_\alpha(n)}$ machine, $\rho(n, \alpha) = a_\alpha + (2^{q_\alpha} - 1)n$, constants a_α, q_α positive integers. \square

Remark 5.11 Now we must estimate the Gödel number of a (reasonably small) clock C_{F_p} that bounds $Q^{F_\alpha(n)}$. Out of the preceding argument we see that the table for the clock has the form:

$$\lfloor [n] \rfloor \lfloor \text{program for } F_\alpha \rfloor \lfloor \text{coupling instructions} \rfloor$$

The only variable portion of it is n . Again we have, for the Gödel numbers $K(\alpha, n)$ of the clocks that bound the $Q^{F_\alpha(n)}$, that, for each n , that they equal $b_\alpha + (2^{q'_\alpha} - 1)n$, q'_α, b_α , positive constants. \square

Remark 5.12 Then the \mathcal{P} -index for the couple

$$\langle Q^{F_\alpha(n)}, C_{F_p} \rangle$$

is a degree-2 polynomial on those linear functions. That is, the series of Gödel numbers for those machines in the representation (set of clock-bounded machines) \mathcal{P} is bounded by a very reasonable function.

However the elements of $\sigma\mathcal{P} \subset \mathcal{M}$ are linearly spaced in \mathcal{M} when this latter set is ordered by the machines' Gödel numbers. For in order to make $\sigma\mathcal{P}$ recursive we write the couples $\langle Q^{F_\alpha(n)}, C_{F_p} \rangle$ as [parameter for the clock][instructions for the clock][instructions for the machine]. The first set of strings may be empty or is easily recognizable; the next one is also a recognizable set (the clock); and then we have an arbitrary machine. The first two sets provide the identification for the members of $\sigma\mathcal{P}$ in \mathcal{M} .

Then the argument presented above shows that those machines are linearly spaced. \square

Corollary 5.13 *The \mathcal{P} -index $\lambda\langle Q^{F_\alpha(n)}, C_{F_p} \rangle$ is a degree-2 polynomial on n . \square*

Corollary 5.14 *The Gödel number $\sigma Q^{F_\alpha(n)}$ is linear on n . \square*

Fast-growing functions “embedded” into $f_{\neg A}$

We obtain a kind of “copy” of a fast-growing function F within $f_{\neg A}$ in such a way that the the images $\sigma F(n), \sigma F(n+1), \dots$, are separated in a “controlled” way within $f_{\neg A}$, that is, $F(n) = f_{\neg A}(p(n)), F(n+1) = f_{\neg A}(p(n+1)), \dots$, where p is a polynomial.

The main lemma

Lemma 5.15 *For no F_α does F_α dominate $f_{\neg A}$.*

Proof of the main lemma

Let the F_α , α a countable ordinal, $\alpha < \epsilon_0$, be the dominating functions in the Kreisel hierarchy [7, 10, 13, 14] in PA. Now recall (Remarks 3.5 and 3.6) that, depending on the unicity or not of ρX , the index (Gödel number) of machine X in PA, it is uniquely defined but for the trivial machine. (See Definition 3.5.) Then:

Definition 5.16 *For any α define the map from F_α to $f_{\neg A}$ given by:*

$$F_\alpha(n) \mapsto f_{\neg A}(\sigma Q^{F_\alpha(n)}) = F_\alpha(n) + 1. \quad \square$$

(We don't need the clock here, as we have added the machines described in Example 3.20 to $\sigma\mathcal{P}$.)

Put $\psi_\alpha(n) = \sigma Q^{F_\alpha(n)}$; it is at worst linear on n . We will be interested in the “peaks” of $f_{\neg A}(m)$ at $m = \psi(n)$, $n = 0, 1, 2, \dots$

Restricted to those values of the variable n ,

$$f_{\neg A}(\psi_\alpha(n)) = F_\alpha(n) + 1.$$

For all $\beta < \alpha$, $f_{\neg A}(n)$ overshoots through $F_\beta(n)$ infinitely many times. Let's see how it is done. We first need:

Lemma 5.17 *For any positive-definite polynomial p and any $\alpha < \epsilon_0$, there is a β , $\alpha < \beta < \epsilon_0$ such that $F_\beta \succ F_\alpha \circ p$.*

Proof: We use this simple characterization for the Kreisel hierarchy [13]:

1. $F_0 = 0$, $F_1 = 2x$.
2. $F_{\beta+1}(x) = F_\beta^{(x)}(1)$.
3. $F_\beta(x) = F_{\beta(x)}(x)$, where the sequence of ordinals $\dots \beta(x) \dots$ converges to the limit ordinal β .

Now suppose that p is a $(n-2)$ -degree positive definite polynomial. Then:

$$F_{\beta+n+1}(x) = F_{\beta+1}^{(x^n)}(1) \succ F_\beta^{(p(x))}(1).$$

As $x^n \succ p(x)$, we are done. For the limit ordinal case, use induction over the sequence of ordinals that converge to limit β . \square

Now to conclude our proof:

- Suppose that $F_\beta \succ f_R$.
- From Lemma 5.17 there is an α such that $F_\alpha \succ F_\beta \circ p$, p a fixed polynomial.
- More precisely, for $x > x_0$, $F_\alpha(x) > F_\beta(p(x))$, or, given a change of variables, $F_\alpha(p^{-1}(y)) > F_\beta(y)$.
- Finally, from our construction, we have that $f_{\neg A}(n) = F_\alpha(p^{-1}(n)) + 1 > F_\beta(n)$, $n > n_0$, and thus our proof. \square

This is of course only valid for $f_{\neg A}$ restricted to the previously given values of $\psi_\alpha(n)$; we aren't interested here in what happens to the in-between values. So we conclude:

Lemma 5.18 *For no $\alpha < \epsilon_0$ does F_α dominate $f_{\neg A}$.*

Proof: By construction and from Proposition 5.9. \square

Remark 5.19 As already spelled out in the Remark that opens this Section, the idea is that $f_{\neg A}$ contains “peaks” that are spaced in a controlled, bounded way. Those peaks overtake infinitely many times the monotonic functions F_α . \square

The consistency result

Recall:

Definition 5.20 *A predicate $P(x_0, \dots, x_n)$ is PA-provably recursive if its characteristic function is PA-provably recursive. \square*

Corollary 5.21 *$A(m, x)$ is PA-provably recursive. \square*

We also need [10]:

Proposition 5.22 *If $P(x, y)$ is PA-provably recursive, given*

$$f_P = \mu_y P(x, y),$$

if $\text{PA} \vdash \forall x \exists y P(x, y)$, then there is an $\alpha < \epsilon_0$ such that $\mathbf{F}_\alpha \succ f_P$. \square

For adequately sound PA, and given Remark 5.2:

Corollary 5.23 $\text{PA} \not\vdash \forall x \exists y \neg A(m, x)$. \square

Corollary 5.24 $\text{PA} \not\vdash \neg[P = NP]^\mathcal{S}$. \square

Remark 5.25 The soundness condition we require was called “arithmetic consistency,” or “arithmetic soundness,” in some previous papers by the authors. See, for instance, [3]. \square

Remark 5.26 Recall that, as already indicated, the result in Corollary 5.24 extends to any fragment \mathcal{T} of ZFC that includes PA and whose consistency strength is measured by a countable, recursive, ordinal. \square

$f_{\neg A}$ is self-similar

Remark 5.27 $f_{\neg A}$ is self-similar in the following sense: intuitively, we can embed h into itself, as in the previous proof. Thus there will be countably infinite many copies of $f_{\neg A}$ within itself. The same is true of any recursive function in whose construction $f_{\neg A}$ appears. \square

Extension to theories beyond PA which are not of ordinal consistency rank

Remark 5.28 We are interested in the further extension of those results to the whole of ZFC. Within ZFC there are functions that are total recursive but not ZFC-provably total recursive (granted that one supposes that every arithmetic formula that is provable in ZFC is true of the standard model). However it isn't immediately clear how we can extend our preceding result to the whole of ZFC, as the Kreisel hierarchy [10, 14] stops at f_{ϵ_0} .

Proposition 5.22 applies to theories for which an “ordinal assignment” measure of consistency-strength is known. Although it applies to many stronger theories than PA, it cannot apply to ZF or ZFC because we have no idea how to measure its consistency strength.

Any use of ZF or ZFC in this context is highly confusing because of this lack of “ordinal measures.” (Personal communication by S. S. Wainer.) \square

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